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On rational functions with Golden Ratio as fixed point

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ON RATIONAL FUNCTIONS WITH GOLDEN RATIO AS FIXED POINT

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Abstract

The existence of an equivalence subset of rational functions with Fibonacci numbers as coefficients and the Golden Ratio as fixed point is proven. The proof is based on two theorems establishing basic relationships underlying the Fibonacci Sequence, Pascal's Triangle and the Golden Ratio. Equations from the two theorems are related to each other and seen to generate the equivalence subset of rational functions. Proof by induction on these equations constitutes the proof of the existence of this subset of rational functions. It is found that this subset of rational functions possesses interesting mathematical properties, particularly that of convergence to the Golden Ratio at the limit. Further investigation showed that this subset of rational functions possesses algebraic structures that would take us into the realms of abstract algebra and complex analysis. The study concludes that the findings are significant as an addition to mathematical knowledge, and as a possible tool for biological research. In this respect, recommendations are made for further research with a view to applications in the sciences and education.

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1 INTRODUCTION

1.1 Rationale of the Study

Echevarria[3, 4] showed that the Golden Ratio induces two alternative mappings of the set of paired Fibonacci numbers into the set of binomial coefficients. In the first mapping, the variant of the Fibonacci Sequence without the initial zero was used. In the second mapping, the zero was retained. The idea that the Golden Ratio induces a mapping of the set of paired Fibonacci numbers into the set of binomial coefficients was the subject of a lecture given by the author, where the first of the above mappings was introduced. The level of mathematics involved in the proofs of the theorems in the author's article is not difficult. Yet, the articles have no precedent in the mathematical literature.

The above-mentioned mappings deal with the Golden Ratio, Pascal's Triangle and the Fibonacci Sequence, three mathematical constructions that date from antiquity. It is known that Pascal's Triangle and Fibonacci Sequence are related mathematically[1]. It is also known that the Golden Ratio and the Fibonacci Sequence are mathematically related in a number of very interesting ways[7]. Ghyka[5] explains that the ancient Greeks, who discovered the Golden Ratio, were familiar with the Fibonacci Sequence as one ramification of numerical operations on the Golden Ratio. What the author showed is that the Golden Ratio, Pascal's Triangle and the Fibonacci Sequence are related not only in a number of pair-wise manners, but in a manner involving all three constructions. In effect, he showed that the Golden Ratio induces two alternative mappings of the set of paired Fibonacci numbers into the set of binomial coefficients.

No mention is made, in the article mentioned[3, 4], regarding possible relationships between the two alternative mappings. However, their similarity in construction, notwithstanding differences in point of departure, suggests the possibility of combining the two mappings algebraically to generate a subset of rational functions. This possibility of further developing a novel idea is the motivation behind the choice of topic. Whenever a new inroad is made in any field of intellectual endeavor, there is always a need to ascertain its full potential. In mathematics, this may result in linkages between hitherto unrelated subjects. In addition, the possibility of applications in other fields of study, no matter how speculative the idea must never be discounted. How could the ancient Greeks have known that the Fibonacci Sequence was to be applied in Computer Science after 2,500 years? It is every generation's task to bequeath to future generations a greater range of intellectual tools and options.

1.2 Theoretical Background

Pascal's Triangle was known to Chinese mathematicians as early as 12^{th} century, to whom it served as a useful computational aid, in the absence of the modern numeral system nowadays taken for granted. It is an infinite triangular array of numbers (see Figure 1) where all numbers on the boundary are equal to 1, and each number in the interior is the sum of the two numbers immediately above it (to the left and to the right).

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & & \\
 & & & & & & & . & & & & & & &
\end{array}
$$

Figure 1: Pascal's Triangle

Pascal's Triangle represents the binomial coefficients, obtained by algebraically expanding the sum of two quantities raised to the power of a whole number[6]. Thus, if x and y are any two quantities, $(x+y)^0 = 1$, giving the topmost number of the triangle; $(x+y)^1 = x+y$, each quantity having 1 as coefficient, yielding the two 1's on the second level; $(x+y)^2 = x^2+2xy+y^2$, the coefficients being the three numbers on the third level of the triangle; and so on. The name is due to James Bernoulli, who rediscovered this triangular array in the 17^{th} century and named it after Blaise Pascal, a fellow mathematician; however, Chinese mathematicians of the 12^{th} century were already familiar with it[2].

The Fibonacci Sequence is the sequence $0, 1, 1, 2, 3, 5, 8, \ldots$ The first two terms are given. All other terms are sums of the two preceding terms[2]. Although the ancient Greeks were familiar with this sequence of numbers, they do not seem to have made much of it. The Fibonacci Sequence was rediscovered by Leonardo of Pisa in the 13^{th} century, while trying to analyze the reproduction pattern of rabbits[9]. Fibonacci is short for "son of Bonaccio" in Italian, and was Leonardo of Pisa's popular name.

The Golden Ratio is a product of early Greek mathematics, which saw the development of the related concepts of ratio and proportion[5]. If p and q are any two natural numbers or positive integers, $\frac{p}{q}$ represents the ratio of p to q. A ratio is the quotient of two numbers or quantities. When two ratios are equated, one obtains a proportion. Thus, $\frac{p}{q} = \frac{r}{s}$ represents a proportion. Proportions of the type $\frac{p}{q} = \frac{q}{r}$ are known as continuous proportions. One particular type of continuous proportion, $\frac{p}{q} = \frac{q}{(p+q)}$, was very special to the ancient Greeks, who named it the Golden Proportion. The Golden Proportion is the simplest of continuous proportions because it has only two unknowns and uses the most basic arithmetic operation, addition. It follows from the Golden Proportion that:

$$\begin{aligned} \frac{q}{p} &= \frac{(p+q)}{q} \\ \frac{q}{p} &= \frac{p}{q} + 1 \\ \left(\frac{q}{p}\right)^2 &= \frac{q}{p} + 1 \end{aligned}$$

The ratio $\frac{q}{p}$ derived from the Golden Proportion was called the Golden Ratio by the ancient Greeks, and its exact value is $\frac{(1 \pm \sqrt{5})}{2}$ obtained by solving $x^2 - x - 1 = 0$ for x, the Golden Ratio $\frac{q}{p}$ being represented by x[2]. The Golden Ratio became the standard of perfection in Greek art and architecture[5]. This tradition was continued by the Romans and carried on into the Middle Ages, where the Golden Ratio is represented in the architecture of the great cathedrals. The great artists and architects of the Renaissance inherited this cultural legacy and passed it on to their heirs in the Modern Age.

It is a mathematical fact that the ratio of a Fibonacci number to the number that precedes it in the sequence approaches the Golden Ratio at the limit of the sequence[7]. It is also a mathematical fact that any Fibonacci number can be represented by a general formula, known as Binet's Formula that incorporates the Golden Ratio[1]. Letting $F(n)$ represent the n^{th} number of the Fibonacci sequence,

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Fibonacci Sequence can be derived from Pascal's Triangle by adding numbers in Pascal's Triangle diagonally[1]. In Figure 2, the diagonal sequence of numbers on the upper right represents the first nine terms of the Fibonacci sequence (excluding the initial zero). They are seen to be the sums of numbers in Pascal's Triangle connected by diagonal lines.

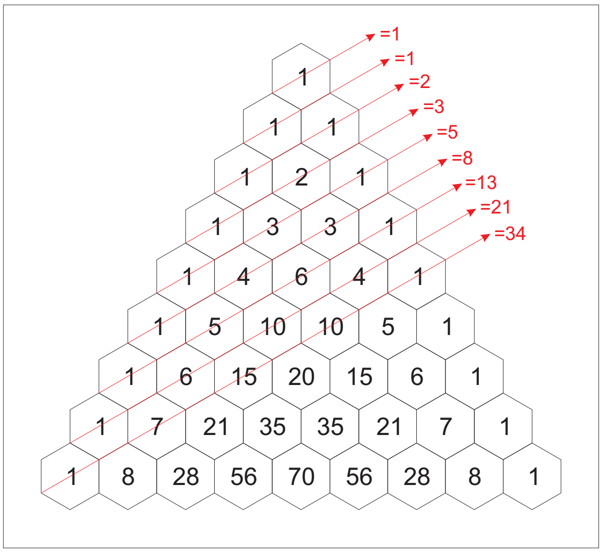

Figure 2: A Pascal's Triangle Fibonacci Sequence Mapping

The Fibonacci Sequence has important applications in Computer Science and Biology, notably in the architecture of plant parts and marine animals[8].

A relation is an equivalence relation if it is reflexive, symmetric, and transitive. If $\sim$ is an equivalence relation on a set S, the set of all elements of S that are equivalent (with respect to $\sim$) to a given element x constitute the equivalence subset of x, denoted by $[x]$.

Consider the functions defined at $x = \varphi$. We can say that;

$$f \sim g \text{ iff } f(\varphi) = g(\varphi)$$

defines an equivalence relation.

Let $[t]$ represents the subset of all functions such that $t(\varphi) = t$. Then $[\varphi]$ is the fixed point subset since $t(\varphi) = \varphi$.

Also, we have;

$$F(2n) + F(2n+1)\varphi = \varphi(\varphi+1)^n$$
$$F(2n+1) + F(2n+2)\varphi = (\varphi+1)^{n+1}$$

So for $t = (\varphi+1)^{n+1}$, $[t]$ contains a line with Fibonacci coefficients $F(2n+1) + F(2n+2)x$ as well as the polynomial $(x+1)^{n+1}$.

And for $t = \varphi(\varphi+1)^n$, $[t]$ contains of course the polynomial $x(x+1)^n$ as well as a straight line with Fibonacci coefficients $F(2n) + F(2n+1)x$.

Take note also that in $[\varphi]$, there is no other line with integer coefficients other than that just found.

1.3 The Problem and its Methodology

The main problem of this study is to prove the existence of an equivalence subset of rational functions on the basis of specific relationships underlying the Golden Ratio, Pascal's Triangle and the Fibonacci Sequence. In particular, the study aims to:

1. Establish basic relationships underlying the Golden Ratio, Pascal's Triangle and the Fibonacci Sequence;
2. Generate theorems that embody these basic relationships;
3. Prove the existence of an equivalence subset of rational functions with the Golden Ratio as fixed point, on the basis of the theorems generated;
4. Identify significant mathematical properties and algebraic structures of this subset of rational functions; and
5. Cite possible applications involving this subset of rational functions.

To prove the existence of the two alternative mappings of paired Fibonacci numbers into binomial coefficients, one can begin by representing the paired Fibonacci numbers as paired coefficients of polynomials in one variable and two coefficients. One can then expand the sum of the variable and 1 (i.e., $x + 1$) to the power of the whole numbers, in ascending order, to obtain polynomials in x with binomial coefficients. Equating the elements of the two sets of polynomials on a one-to-one basis, it can be shown that the equations hold if x is the solution of the equation $x^2 - x - 1 = 0$[4].

Two alternative mappings will be generated on the basis of two variants of the Fibonacci Sequence, one variant beginning with 0 and the other variant disregarding the initial 0. The two variants have no paired Fibonacci numbers in common. What they have in common is the fact that only one pair of Fibonacci numbers from each variant corresponds to a given binomial expansion. Expressing these correspondences as equations involving polynomials in one variable and two coefficients permits algebraic operations with a view to characterizing the Golden Ratio as a fixed point of a subset of rational functions. Figure 3 shows the flow diagram of the procedure.

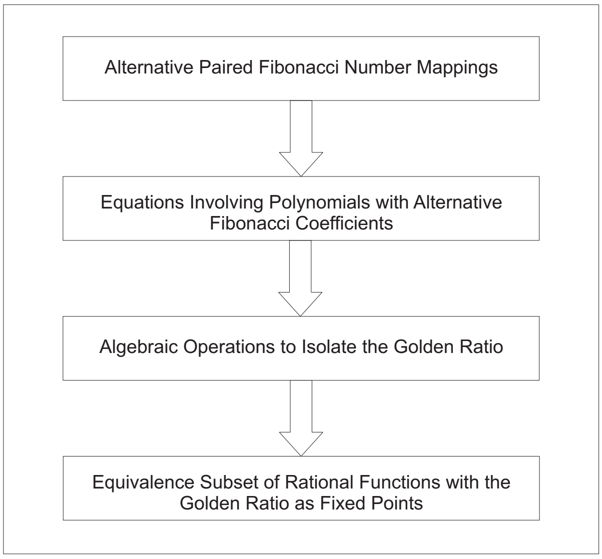

Figure 3: Flow Diagram of Research Procedure

2 ALTERNATIVE FIBONACCI MAPPINGS AND AN EQUIVALENCE SUBSET OF RATIONAL FUNCTIONS

2.1 Two Mappings of Paired Fibonacci Numbers into the Binomial Coefficients

Two theorems, each one establishing an alternative mapping of paired Fibonacci numbers into the binomial coefficients, and each mapping being induced by the Golden Ratio, are proven in this section. In the first theorem, the variant of the Fibonacci Sequence without the initial zero is utilized. In the second theorem, the variant with initial zero is utilized. The theorems are due to Echevarria (1997:89-94).

Theorem 1. *Let $F(n)$ be the Fibonacci Sequence defined by*

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \forall n \geq 2 \text{ and } x = \varphi = \frac{1+\sqrt{5}}{2}$$

then

$$\xi : \{(F(n) \centerdot F(n+1))\} \rightarrow n^{th} \text{ - row of Pascal's Triangle.}$$

such that as $x = \varphi$,

$$F(n) + F(n+1)x = (x+1)^n \text{ is one-to-one.}$$

Proof. We proceed by induction on n Assume true for $n = 1$.

$$F(n) + F(n+1)x = 1 + 1(x) = 1 + x = (x+1)^1 = x + 1$$

Assume it is true for all $n = k$, so that;

$$F(n) + F(n+1)x = x^k + m_1 x^{k-1} + m_2 x^{k-2} + \cdots + m_2 x^2 + m_1 x + 1 = (x+1)^k$$

Show true for $n = k + 1$.

$$\begin{aligned}
(F(n) + F(n+1)) + (F(n+1) + F(n+2))x \\
&= x^{k+1} + (m_1+1)x^k + (m_1+m_2)x^{k-1} + \cdots + (m_1+m_2)x^2 \\
&+ (m_1+1)x + 1 \\
\rightarrow \cancel{(F(n) + F(n+1)x)} + (F(n+1) + F(n)x + F(n+1)x) \\
&= \cancel{(x^k + m_1 x^{k-1}} + \cancel{m_2 x^{k-2}} + \cdots + \cancel{m_2 x^2 + m_1 x + 1)} \\
&+ (x^{k+1} + m_1 x^k + m_2 x^{k-1} + \cdots + m_1 x^2 + x) \\
\rightarrow F(n+1) + F(n)x + F(n+1)x &= (x^{k+1} + m_1 x^k + m_2 x^{k-1} + \cdots + m_1 x^2 + x) \\
&= x(x^k + m_1 x^{k-1} + m_2 x^{k-2} + \cdots + m_2 x^2 + m_1 x + 1) \\
&= x(F(n) + F(n+1)x) \\
F(n+1) + \cancel{F(n)x} + F(n+1)x &= \cancel{F(n)x} + F(n+1)x^2 \\
F(n+1)x^2 - F(n+1)x - F(n+1) &= 0 \\
\rightarrow x^2 - x - 1 &= 0
\end{aligned}$$

Solving the quadratic equation above will give us;

$$x = \frac{1 \pm \sqrt{5}}{2} \rightarrow x = \varphi = \frac{1 + \sqrt{5}}{2}$$

This concludes the proof. □

Since the proof of the next theorem follows practically the same procedure as the proof just given, it can be presented in concise form.

Theorem 2. *Let $F(n)$ be the Fibonacci Sequence defined by*

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \forall n \geq 2 \text{ and } x = \varphi = \frac{1+\sqrt{5}}{2}$$

then

$$\xi : \{(F(n) \centerdot F(n+1))\} \rightarrow n^{th} \text{ -row of Pascal's Triangle.}$$

such that as $x = \varphi$,

$$F(n) + F(n+1)x = x(x+1)^n \text{ is one-to-one and onto.}$$

Proof. The proof is an induction on n. Show true for $n = 0$.

$$F(n) + F(n+1)x = 0 + 1x = x = x(x+1)^0 = x(1)$$

Assume it is true for all $n = k$, so that;

$$F(n) + F(n+1)x = x(x^k + m_1 x^{k-1} + m_2 x^{k-2} + \cdots + m_2 x^2 + m_1 + 1)$$

Show true for $n = k + 1$.

$$\begin{aligned}
(F(n) + F(n+1)) + (F(n+1) + F(n+2))x \\
&= x(x^{k+1} + (m_1+1)x^k + (m_1+m_2)x^{k-1} + \cdots \\
&+ (m_2+m_1)x^2 + (m_1+1)x + 1) \\
\rightarrow \cancel{(F(n) + F(n+1)x)} + (F(n+1) + (F(n+1) + F(n))x) \\
&= \cancel{x(x^k + m1x^{k-1}} + \cancel{m_2xk - 2+} \cdots \cancel{+m_2x^2 + m_1 + 1)} \\
&+ x(x^{k+1} + m_1x^k + m_2x^{k-1} + \cdots + m_2x^3 + m_1x^2 + x) \\
\rightarrow F(n+1) + F(n)x + F(n+1)x &= x(x(x^k + m_1x^{k-1} + m_2x^{k-2} + \cdots + m_1x + 1)) \\
F(n+1) + \cancel{F(n)x} + F(n+1)x &= x(F(n+1)x + F(n)) \\
&= F(n+1)x^2 + \cancel{F(n)x}
\end{aligned}$$

$$\begin{aligned}
F(n+1)x^2 - F(n+1)x - F(n+1) &= 0 \\
\rightarrow x^2 - x - 1 &= 0
\end{aligned}$$

Solving the quadratic equation above will give us;

$$x = \frac{1 \pm \sqrt{5}}{2} \rightarrow x = \varphi = \frac{1+\sqrt{5}}{2}$$, the Golden Ratio by definition.

Unlike the first mapping, the entire set of binomial coefficients is covered by the second mapping. This characteristics makes the mapping an onto mapping. □

2.2 An Equivalence Subset of Rational Functions with the Golden Ratio as Fixed Point

Consider a subset of rational functions in the variable x of the form $\frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}$, where x is any real number. The following theorem shows that, for specific values of $F(n)$, $F(n+1)$, and $F(n+2)$ (in fact, infinite in number), this subset of rational functions forms an infinite equivalence subset, equivalence being defined by a fixed point relationship with the Golden Ratio.

Theorem 3. *Let $F(n)$ be the Fibonacci Sequence defined by*

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \forall n \geq 2 \text{ and } \varphi = \frac{1+\sqrt{5}}{2}$$

then

(i) $x = \dfrac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}$, $x = \varphi$, is an equivalence subset of rational function with Golden Ratio as fixed point;

(ii) $lim_{n\to\infty} \dfrac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x} = \varphi$

Proof(i). From Theorems 1 and 2, the two systems of equations given below hold when $x = \varphi$.

(i)

$$\begin{aligned} 1 + 1x &= x + 1 \\ 2 + 3x &= (x+1)^2 \\ &\vdots \\ F(n) + F(n+1)x &= (x+1)^n \end{aligned} \tag{1}$$

(ii)

$$\begin{aligned} 0 + 1x &= x(1) \\ 1 + 2x &= x(x+1) \\ &\vdots \\ F(n+1) + F(n+2)x &= x(x+1)^n \end{aligned} \tag{2}$$

where $F(n+2) = F(n) + F(n+1)$

Dividing the n^{th} equation of the second system by the n^{th} equation of the first system, we obtain:

$$\frac{x\cancel{(x+1)^n}}{\cancel{(x+1)^n}} = \frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}$$
$$x = \frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}$$

There remains to show that $x = \varphi$. Dividing the $(n+1)^{th}$ of (1) by the n^{th} of (2) yields:

$$\frac{(x+1)^{n+1}}{x(x+1)^n} = \frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x} = \frac{x+1}{x}$$

But

$$
\begin{aligned}
x &= \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \\
\rightarrow x &= \frac{x+1}{x} \\
x^2 - x - 1 = 0 & \\
\rightarrow x &= \frac{1 \pm \sqrt{5}}{2}, \text{ in particular } x = \frac{1+\sqrt{5}}{2} = \varphi
\end{aligned}
$$

This concludes the proof. □

Proof(ii). As $n \rightarrow \infty$, using the Binet's Formula for $F(n)$:

$$
\begin{aligned}
F(n) &= \frac{x^n - y^n}{x-y} \text{ where } x = \frac{1+\sqrt{5}}{2}, y = \frac{1-\sqrt{5}}{2} \\
\rightarrow x + y &= 1 \text{ and } x - y = \sqrt{5} \\
\lim_{n\rightarrow\infty} \frac{F(n+1)}{F(n)} &= \lim_{n\rightarrow\infty} \frac{x^{n+1} - y^{n+1}}{x^n - y^n} = \lim_{n\rightarrow\infty} \frac{x^{n+1}}{x^n} = x
\end{aligned}
$$

Since $|y| < 1$ so as $n \rightarrow \infty$, $y^n \rightarrow 0$.

Hence,

$$
\begin{aligned}
&\lim_{n\rightarrow\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \lim_{n\rightarrow\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \cdot \frac{F(n+1)}{F(n+1)} \\
&\rightarrow \lim_{n\rightarrow\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \cdot \frac{\frac{1}{F(n+1)}}{\frac{1}{F(n+1)}} = \lim_{n\rightarrow\infty} \frac{1 + \frac{F(n+2)x}{F(n+1)}}{\frac{F(n)}{F(n+1)} + x} \\
&= \frac{1 + x \cdot x}{\frac{1}{x} + x} = \frac{1+x^2}{\frac{1+x^2}{x}} = 1 + x^2 \cdot \frac{x}{1+x^2} = x = \frac{1+\sqrt{5}}{2} = \varphi.
\end{aligned}
$$

This concludes the proof. □

Corollary. *The smallest positive Fibonacci triples satisfying*

$$\varphi = \frac{F(n+1)+F(n+2)\varphi}{F(n)+F(n+1)\varphi} \text{ is (1,1,2).}$$

Proof. Immediate from Theorem 3. By inspection, the consecutive Fibonacci triples forming the equivalence subset of rational functions are: (1,1,2), (2,3,5), (5,8,13), ..., $(F(n), F(n+1), F(n+2))$ which yields $F(n) = 1, F(n+1) = 1, F(n+2) = 2$ as the smallest positive Fibonacci triples. □

In summary, the fixed point subset, $[\varphi]$ of function defined by $f(\varphi) = \varphi$ contains the line $y = x$ and Fibonacci coefficients $F(0) + F(1)x$.

In addition, the equivalence subset contains all rational functions of the form:

$$y = \frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}.$$

The rational functions $y = \frac{F(n+1) + F(n+2)x}{F(n) + F(n+1)x}$ has two fixed points, the two solutions of $x^2 = x + 1$, namely, $\varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi_1 = \frac{1-\sqrt{5}}{2}$.

Geometrically, these hyperbolas have a vertical asymptote at $x = -\frac{F(n)}{F(n+1)}$ and a horizontal asymptote at $y = \frac{F(n+2)}{F(n+1)}$. Further, as n increases, the horizontal aymptote approaches $y = \varphi$ and the vertical asymptote tends to $x = -\frac{1}{\varphi} = \varphi_1$.

At this juncture, it has been proven that there exists an equivalence subset of rational functions having the Golden Ratio as a fixed point. It has been demonstrated that this subset of rational functions incorporates sets of Fibonacci numbers, that this equivalence subset has an infinite number of members, and that there is a total of two fixed points. By utilizing analytical methods from Calculus, other properties can be found relating to this subset of rational functions to the Golden Ratio, in particular, the absence of extreme values and convergence to the Golden Ratio.

3 ALGEBRAIC PROPERTIES AND STRUCTURES

It is interesting to note that the equivalence subset of rational functions is related to the Golden Ratio in more ways than can be determined by the use of algebraic methods. At least two such properties, namely, the absence of extreme values and convergence to the Golden Ratio, can be identified and to be discussed in this chapter. Further, it will be shown that this equivalence subset of rational functions possesses an algebraic structure, namely, the formation of a commutative semigroup of rational functions under the operation of composition of functions.

3.1 Extreme Values

To verify if the equivalence subset of rational functions has extreme values, one sets:

$$\frac{d}{dx}\left(\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)=0$$

$$\frac{F(n)+F(n+1)x(F(n+2))-(F(n+1)+F(n+2)x(F(n+1))}{(F(n)+F(n+1)x)^2}=0$$

$$\frac{F(n)\cdot F(n+2)-(F(n+1))^2}{(F(n)+F(n+1)x)^2}=0$$

If $F(n)\cdot F(n+2)-(F(n+1))^2\neq 0$ the equation will not hold for finite values of n. Recall that it is always true that $F(n)\cdot F(n+2)-(F(n+1))^2=\pm 1$. Therefore, the equivalence subset of rational functions has **no extreme values** for finite values of $F(n), F(n+1)$, and $F(n+2)$.

3.2 Convergence of Real-Valued Rational Functions

Since each rational function of the equivalence class is uniquely determined by the values of $F(n), F(n+1)$, and $F(n+2)$ corresponding to it, one can construct a sequence of rational functions in terms of increasing values for $F(n), F(n+1)$, and $F(n+2)$. The question is what happens to this sequence as $F(n), F(n+1)$, and $F(n+2)$ tend to infinity.

As what has been proven in Theorem 3, the sequence of rational functions converges to the Golden Ratio as the values of the Fibonacci numbers in the function tend to infinity.

3.3 A Commutative Semigroup of Rational Functions

It will be shown in this section that the equivalence subset of rational functions forms a commutative semigroup under the operation of composition of functions.

In addition, the infinite number of elements is generated by a single element under the same operation, converging to a limit.

The general form of the rational function is $\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$ where $F(n), F(n+1)$, and $F(n+2)$ are consecutive Fibonacci numbers and x represents any real number other than $-\frac{F(n)}{F(n+1)}$. To accomplish our objective, it will suffice to show that this subset of rational functions is closed, associative and commutative under composition of functions, and that it is contained a cyclic subgroup that converges to a limit.

Closure, Associativity and Commutativity

Associativity follows immediately from the fact that composition of real valued functions is associative. We now show closure and commutativity simultaneously. Given two functions $\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$ and $\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}$, composition following the order of appearance of the functions gives:

$$
\begin{aligned}
&\frac{F(n+1)+F(n+2)\cdot\left(\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}\right)}{F(n)+F(n+1)\cdot\left(\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}\right)}\\
=&\frac{F(n+1)\cdot(F(m)+F(m+1)x)+F(n+2)\cdot(F(m+1)+F(m+2)x)}{F(n)\cdot(F(m)+F(m+1)x)+F(n+1)\cdot(F(m+1)+F(m+2)x)}\\
=&\frac{(F(n+1)\cdot F(m)+F(n+2)\cdot F(m+1))+(F(n+1)\cdot F(m+1)+F(n+2)\cdot F(m+2))x}{(F(n)\cdot F(m)+F(n+1)\cdot F(m+1))+(F(n)\cdot F(m+1)+F(n+1)\cdot F(m+2))x}\\
=&\frac{(F(n+1)\cdot F(m)+F(n)\cdot F(m+1)+F(n+1)\cdot F(m+1))}{(F(n)\cdot F(m)+F(n+1)\cdot F(m+1))+(F(n)\cdot F(m+1)+F(n+1)\cdot F(m)+F(n+1)\cdot F(m+1))x}\\
+&\frac{(F(n+1)\cdot F(m+1)+F(n)\cdot F(m)+F(n)\cdot F(m+1)+F(n+1)\cdot F(m)+F(n+1)\cdot F(m+1))x}{(F(n)\cdot F(m)+F(n+1)\cdot F(m+1))+(F(n)\cdot F(m+1)+F(n+1)\cdot F(m)+F(n+1)\cdot F(m+1))x}\\
=&\frac{F(n+1)^*+F(n+2)^*x}{F(n)^*+F(n+1)^*x}
\end{aligned}
$$

where $F(n)^* = F(n)\cdot F(m)+F(n+1)\cdot F(m+1)$; $F(n+1)^* = F(n)\cdot F(m+1)+F(n+1)\cdot F(m)+F(n+1)\cdot F(m+1) = F(n+1)\cdot F(m)+F(n+2)\cdot F(m+1)$; $F(n+2)^* = F(n)^*+F(n+1)^*$. That $F(n)^*$ and $F(n+1)^*$ are consecutive Fibonacci numbers follows from the number theoretic result that $F(n+m) = F(n)F(m+1)+F(m)F(n-1)$, where $F(i)$ is the i^{th} Fibonacci number. Reversing the order of composition gives:

$$
\begin{aligned}
&\frac{F(m+1)+F(m+2)\cdot\left(\dfrac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)}{F(m)+F(m+1)\cdot\left(\dfrac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)}\\
=&\frac{F(m+1)\cdot(F(n)+F(n+1)x)+F(m+2)\cdot(F(n+1)+F(n+2)x)}{F(m)\cdot(F(n)+F(n+1)x)+F(m+1)\cdot(F(n+1)+F(n+2)x)}\\
=&\frac{(F(m+1)\cdot F(n)+F(m+2)\cdot F(n+1))+(F(m+1)\cdot F(n+1)+F(m+2)\cdot F(n+2))x}{(F(m)\cdot F(n)+F(m+1)\cdot F(n+1))+(F(m)\cdot F(n+1)+F(m+1)\cdot F(n+2))x}\\
=&\frac{F(n+1)^*+F(n+2)^*x}{F(n)^*+F(n+1)^*x}
\end{aligned}
$$

The equivalence subset of rational functions is, therefore, closed, associative and commutative under the composition of functions, which makes it a commutative semigroup under the operation.

Absence of an Identity Element

Letting $g(x)=\dfrac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$, an identity element under composition of functions should yield $fg(x)=gf(x)=g(x)$. However, it is known that this fixed point relationship does not generally hold, but only when $\dfrac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}=\varphi$.

The Generator

The element $\dfrac{(1+x)}{x}$ where $F(n), F(n+1)$, and $F(n+2)$ are the first three Fibonacci numbers **0**, **1** and **1**, generates all the others under composition of functions. The second element (corresponding to the second, third and fourth Fibonacci numbers) is $\dfrac{(1+2x)}{(1+x)}$ which is $\dfrac{1+\dfrac{(1+x)}{x}}{\dfrac{(1+x)}{x}}$.

Given any arbitrary $\dfrac{(F(n+1)+F(n+2)x)}{F(n)+F(n+1)x}$, replacing x with $\dfrac{(1+x)}{x}$ yields $\dfrac{(F(n+2)+(F(n+1)+F(n+2))x)}{F(n+1)+F(n+2)x}$ which is the element immediately following the arbitrary element. Thus, $\dfrac{(1+x)}{x}$ generates all the rational functions in the semigroup under composition of functions.

4 SUMMARY, FINDINGS, CONCLUSIONS AND RECOMMENDATIONS

4.1 Summary

In this study, the writer set out with the main objective of proving the existence of an equivalence subset of rational functions with Fibonacci coefficients, and the Golden Ratio as fixed point. The existence of the equivalence subset was suggested by[3, 4] on his recent work on the relationships underlying the Fibonacci sequence, Pascal's Triangle and the Golden Ratio. Both the novelty of the underlying relationships and the possibility of applications in the biological sciences made the study particularly significant. The proof hinged on two recent theorems relating to Fibonacci sequence, Pascal's Triangle and the Golden Ratio, and involved algebraic operations on equations derived in the proofs of the theorems. Having accomplished the main objective, the writer set out to identify significant mathematical properties of the equivalence subset of rational functions. Further investigation of the mathematical properties was made. It was found that the subset of rational functions possesses an algebraic structure, namely, the formation of commutative semigroup of rational functions under the operation of composition of functions.

4.2 Findings

The main finding of this study is the existence of an equivalence subset of rational functions with Fibonacci coefficients, and with the Golden Ratio as fixed point. It was found that this subset of rational functions possesses interesting mathematical properties and algebraic structure, as follows:

- The existence of a total of two fixed points in the equivalence subset.
- The incorporation of unique sets of Fibonacci numbers in each function of the equivalence subset.
- The infinite number of members in the equivalence subset.
- The absence of extreme values for any given rational function.
- The convergence of the sequence of a real valued rational functions to the Golden Ratio as the Fibonacci numbers in the function tends to infinity.
- The equivalence subset of rational functions forms a commutative semigroup under composition of functions.

- The infinite number of elements in the semigroup is generated by a single element, $\dfrac{1+x}{x}$ under composition of functions.
- The solution set of an infinite system of equations where the functions on the left are linear with consecutive Fibonacci pairs as coefficients, and those on the right are binomial expansions of $x+1$ is a set of fixed points of two subsets of rational functions which differ by virtue of a well known fact about Fibonacci numbers, namely, that if $F(n), F(n+1)$, and $F(n+2)$ are consecutive Fibonacci numbers, then $F(n) \cdot F(n+2) - (F(n+1))^2 = \pm 1$.

4.3 Conclusions

It is the writer's conclusion that this study is important for a number of reasons. First, it is a new addition to mathematical knowledge. Even if this contribution turns out to be a mathematical dead end, the knowledge that such is the case is additional knowledge just the same. The writer is presently motivated to contact mathematicians and scientists who might find the study of significance in their work, which brings us to the second reason why the study is important, namely, the possibility of applications in biological sciences. Thirdly, this study is important since it is easily adaptable for use as educational material at the undergraduate level, particularly for mathematics majors. In this respect, there is the added bonus that some student may develop it further.

4.4 Recommendations

In view of the writer's reasons for the importance of this study, it is highly recommended that contacts with scientists and mathematicians be initiated, once the bound copies are submitted, with a view to possible applications in science and further mathematical development. In the same vein, it is recommended that the suitability of selected parts of this study for a number of mathematical courses offered at Rochester Institute of Technology be studied.

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